CONCERNING A TURBULENT DIFFUSION IN A STREAM WITH A TRANSVERSE GRADIENT OF VELOCITY

(O TURBULENTNOI DIFFUSII V POTOKE S POPERECHNYM GRADIENTOM SKOROSTI)

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E.A. NOVIKOV
(Moscow)

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We will consider the diffusion of an impurity from an instantaneous point source in a horizontal stream, the velocity of which varies with height in accordance with a linear law.

One of the most urgent problems of the theory of turbulent diffusion is the description of the behavior of a diffusing impurity in a stream with a velocity gradient. We shall consider the diffusion of an impurity from an instantaneous point source in a horizontal stream, the velocity components of which depend linearly on height as follows:

$$U_m(x_3, t) = v_m(t) + a_m(t)x_3 \qquad (m = 1, 2) \tag{1}$$

The vertical axis x_3 can be considered in this case as the principal axis of the turbulent diffusion tensor [1]. A semiempirical equation of turbulent diffusion, taking into account gravitational sedimentation of particles, can be written in the following form*:

$$\frac{\partial q}{\partial t} + [v_m(t) + a_m(t)x_3]\frac{\partial q}{\partial x_m} - w\frac{\partial q}{\partial x_3} = K_{\mu\nu}(t)\frac{\partial^2 q}{\partial x_\mu \partial x_\nu} \tag{2}$$

Here q = concentration of the impurity, w = steady-state velocity of particle falling in a quiescent medium, $K_{\mu\nu}(t)$ are the components of the tensor of turbulent diffusion, assumed independent at the coordinates, with $K_{m3}(t) = K_{3m}(t) = 0$.

The initial condition corresponding to the instantaneous point source located at a height h, has the form:

$$q|_{t=0} = Q\delta(x_1)\delta(x_2)\delta(x_3 - h) \tag{3}$$

* In equation (2) and in the following, the Latin indices m, n run through 1 and 2, whereas the Greek indices μ, ν run through 1, 2 and 3. Repeated indices denote summation.

where Q is the total amount of impurity.

The boundary condition consists of the condition that the concentration q become zero at infinity.

We shall introduce new variables, which will allow us to eliminate purely kinematical terms from equation (2):

$$y_m = x_m - \int_0^t v_m(\tau)\,d\tau - x_3\int_0^t a_m(\tau)\,d\tau - w\int_0^t (t-\tau)\,a_m(\tau)\,d\tau, \qquad y_3 = x_3 - h + wt \quad (4)$$

These variables emerge naturally in the solution of the kinematic problem, i.e. the solution of equation (2) without the right-hand side.

Using the new variables, equation (2) and the initial condition (3) assume the following form:

$$\frac{\partial q}{\partial t} = S_{\mu\nu}(t)\frac{\partial^2 q}{\partial y_\mu \partial y_\nu}, \qquad q\,|_{t=0} = Q\delta(y_1)\,\delta(y_2)\,\delta(y_3) \quad (5)$$

where

$$S_{mn}(t) = K_{mn}(t) + K_{33}(t)\left(\int_0^t a_m(\tau)\,d\tau\right)\left(\int_0^t a_n(\tau)\,d\tau\right)$$

$$S_{m3}(t) = S_{3m}(t) = -K_{33}(t)\int_0^t a_m(\tau)\,d\tau, \qquad S_{33}(t) = K_{33}(t) \quad (6)$$

We apply a two-sided Laplace transformation with respect to all three variables y_μ in equation (5). For the result we obtain:

$$\frac{dq^\circ}{dt} = S_{\mu\nu}(t)\,\alpha_\mu\alpha_\nu q^\circ, \qquad q^\circ\,|_{t=0} = Q \quad (7)$$

Here α_μ is the transformation parameter with regard to the variable y_μ, q° is the transform of the function q.

The solution of this problem has the form:

$$q^\circ = Q\exp\{T_{\mu\nu}(t)\,\alpha_\mu\alpha_\nu\} \qquad \left(T_{\mu\nu}(t) = \int_0^t S_{\mu\nu}(\tau)\,d\tau\right) \quad (8)$$

The inverse Laplace transform is

$$q = \frac{Q}{(2\pi i)^3}\int\!\!\int\!\!\int_{\sigma-i\infty}^{\sigma+i\infty} \exp\{T_{\mu\nu}(t)\,\alpha_\mu\alpha_\nu + \alpha_\mu a_\mu\}\,d\alpha_1\,d\alpha_2\,d\alpha_3 \quad (9)$$

The parameter σ may, in our case, be put equal to zero. We shall replace α_μ by $i\beta_\mu$.

Then

$$q = \frac{Q}{(2\pi)^3}\int\!\!\int\!\!\int_{-\infty}^{\infty} \exp\{-T_{\mu\nu}(t)\,\beta_\mu\beta_\nu + i\beta_\mu a_\mu\}\,d\beta_1\,d\beta_2\,d\beta_3 \quad (10)$$

and after some rather simple computations we obtain finally

$$q = \frac{Q}{(4\pi)^{3/2}\sqrt{\mathrm{Det}\,(T)}} \exp\left\{-\frac{1}{4}\, T_{\mu\nu}^{-1}(t)\, y_\mu y_\nu\right\} \tag{11}$$

Here $T_{\mu\nu}^{-1}(t)$ is the matrix inverse to $T_{\mu\nu}(t)$, and $\mathrm{Det}(T)$ is the determinant of the matrix $T_{\mu\nu}(t)$.

In order that the integral (10) have meaning and that it be equal to expression (11), it is necessary that the following inequalities be fulfilled:

$$T_{11} > 0, \qquad T_{11}T_{22} - (T_{12})^2 > 0, \qquad \mathrm{Det}\,(T) > 0 \tag{12}$$

(T_{11} and T_{22} are arbitrarily selected diagonal elements of the matrix). Inequalities (12) impose definite limitations on the matrix $S_{\mu\nu}(t)$ in equation (5), and in addition they assure positive definiteness of the quadratic form $T_{\mu\nu}^{-1}(t)\, y_\mu y_\nu$. The last circumstance ensures that the concentration q vanishes at infinity.

Equations (11), (8), (6) and (4) give the solution of the proposed problem. We shall consider in greater detail the important case of a stationary stream, in which case the parameters v_m and a_m in equation (1) may be considered stationary. The components of the turbulent diffusion tensor will be also considered stationary. The horizontal coordinate axis in the case of a stationary stream may be so selected that the parameter a_2 in formula (1) is zero. Coordinate axes selected in this way are also the principal axes of the turbulent diffusion tensor [1]. We denote the corresponding coefficients of turbulent diffusion by k_1, k_2 and k_3. Inasmuch as these coefficients are positive, it is easy to check that the inequalities (12) are fulfilled.

In view of the preceding remarks, equation (11) can be reduced to the following form:

$$q = \frac{Q}{(4\pi t)^{3/2}\sqrt{k_1^* k_2 k_3}} \times$$
$$\times \exp\left\{-\frac{[x_1 - v_1 t - {}^1/_2\, a_1 t\,(x_3 + h)]^2}{4k_1^* t} - \frac{(x_2 - v_2 t)^2}{4k_2 t} - \frac{(x_3 - h + wt)^2}{4k_3 t}\right\} \tag{13}$$

where

$$k_1^* = k_1 + \frac{1}{12}\, k_3 a_1^2 t^2 \tag{14}$$

This equation can be used for the description of the dispersion of an impurity in a free atmosphere with a vertical gradient of wind velocity.

Analysis of formula (13) leads to the following basic conclusions.

1. After the lapse of a certain time when the inequality

$$k_3 a_1^2 t^2 \gg 12 k_1 \quad (15)$$

begins to hold true, one may neglect the coefficient of turbulent diffusion k_1.

2. After the lapse of the above time, the concentration at the center of the diffusing cloud begins to decay with time as $t^{-5/2}$ (instead of $t^{-3/2}$, in the absence of the velocity gradient).

3. Lines of equal concentration in the plane $x_2 = v_2 t$ are ellipses which, turning, are strongly elongated in the horizontal direction. The ratio between the horizontal and vertical axes of the ellipse for long t become proportional to $a_1 t$. Thus, the frequent observation that clouds of diffusing material are elongated in a horizontal direction can be explained by the presence of a vertical wind gradient instead of assuming that $k_1 > k_3$. The last explanation could be valid when there is no change of wind velocity with height.

BIBLIOGRAPHY

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